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Overall Solution-Difference Bounds on the Effects of Material Inhomogeneities

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Abstract. In this work variational bounds are developed on the difference between the solutions of two boundary-value problems involving linearly elastic material microstructure. Both problems have the same external geometry, boundary conditions and loading, however, one possesses no microstructure, i.e., it is homogeneous, while the other is "perturbed" with inhomogeneities. The bounds developed are solely in terms of the solution to the easier homogeneous material problem, the microstructure of the inhomogeneous problem and the given loading data. There are no unknown constants in the bounds. Furthermore, no calculation of the harder, typically intractable, inhomogeneous material problem is necessary. The bounds developed are obtained under no assumptions on the character of the microstructure of the inhomogeneous material problem, other than it be pointwise positive-definite, as well as under no assumptions on the external loading and geometry.

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1. Introduction

In many modern engineering designs, materials with a highly complex microstructure are in use. A relatively common type of such materials is that formed by a homogeneous matrix embedded with a particulate material (Figure 1). The analysis of microheterogeneous materials is not a recent development. Historically, due to the extreme difficulty of solving boundary-value problems with highly variable coefficients, the usual interest has been in bounding or estimating the aggregate's macroscopic response. Works dating back at least to Maxwell [1] (1873), Voigt [2] (1889), Lord Rayleigh [3] (1892) and Reuss [4] (1929) dealt with determining overall macroscopic behavior of materials consisting of a binding matrix containing distributions of particles. In solid mechanics, the classical goal has usually been to determine a relation between averages, a so-called "effective" property, which is a macroscopic scale linear elasticity tensor, $\mathbb{E}^*$, defined via $\langle \boldsymbol{\sigma} \rangle_\Omega = \mathbb{E}^* : \langle \boldsymbol{\epsilon} \rangle_\Omega$, where $\langle \cdot \rangle_\Omega \stackrel{\text{def}}{=} (1/|\Omega|) \int_\Omega \cdot \, \mathrm{d}\Omega$, and where $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the stress and strain tensor fields within a statistically representative volume element with volume $|\Omega|$. The reader is referred to reviews found in [5].

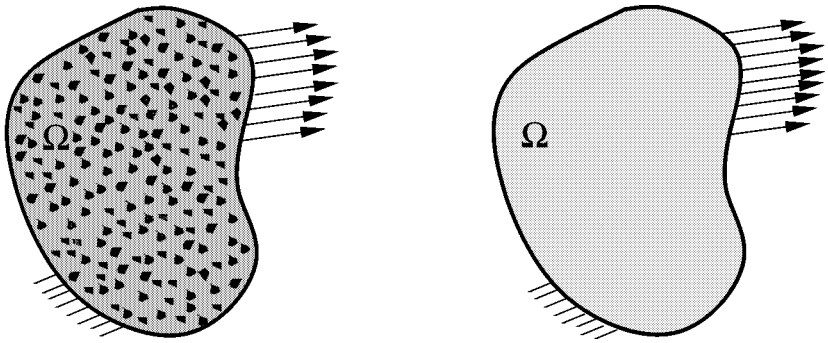

Figure 1. A structure with microheterogeneities and a homogeneous structure.

Effective properties are not the objective of this work. Whenever information about the effects of inhomogeneities on the internal microfields is of interest, estimates or bounds on the effective property are of little value. *In such cases estimates or bounds on microfield quantities are necessary.* In this work, in order to characterize the effects of the inhomogeneities onto the material, bounds are developed on the *difference between the solutions* to two boundary-value problems involving linearly elastic microstructure. Both boundary-value problems have the same external geometry, boundary conditions and loading. However, one possesses a uniform microstructure, while the other is "perturbed" with inhomogeneities (Figure 1). As will be shown, the bounds developed in this work are solely in terms of the solution to the easier homogeneous material problem, the microstructure of the inhomogeneous problem and the given loading data. There are no unknown constants in the bounds. Furthermore, no calculation of the computationally expensive heterogeneous problem is necessary. The bounds are obtained under no assumptions on the character of microstructure of the heterogeneous problem, other than it be pointwise positive-definite, as well as under no assumptions on the external loading and geometry.

2. Governing Equations

A structure which occupies an open bounded domain in $\Omega \in \mathcal{R}^3$ is considered. Its boundary is denoted by $\partial\Omega$. The body is in static equilibrium under the action of body forces $\mathbf{f}$ and surface tractions $\mathbf{t}$. The boundary $\overline{\partial\Omega} = \overline{\Gamma_u \cup \Gamma_t}$ consists of a part Γ_u and a part Γ_t on which displacements and tractions are respectively prescribed. The data are assumed to be such that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{t} \in \mathbf{L}^2(\Gamma_t)$, but less smooth data can be considered without complications. The mechanical properties of the material are characterized by a (regular) spatially constant inhomogeneity-free elasticity tensor $\mathbb{R}$ and a spatially nonconstant (perturbed) elasticity tensor

$\mathbb{E}(\mathbf{x})$, both of which are in $\mathcal{R}^{3^2\times 3^2}$, and whose components satisfy the following symmetry and boundedness conditions: $\forall \boldsymbol{\epsilon} \in \mathcal{R}^{3\times 3}$, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{T}}$, $\alpha^{+,\mathrm{R}}\boldsymbol{\epsilon} : \boldsymbol{\epsilon} \geqslant \boldsymbol{\epsilon} : \mathbb{R} : \boldsymbol{\epsilon} \geqslant \alpha^{-,\mathrm{R}}\boldsymbol{\epsilon} : \boldsymbol{\epsilon}$, $\infty > \alpha^{+,\mathrm{R}}, \alpha^{-,\mathrm{R}} > 0$, where $R_{ijkl} = R_{jikl} = R_{ijlk} = R_{klij}$, $1 \leqslant i, j, k, l \leqslant 3$, and $\alpha^{+}\boldsymbol{\epsilon} : \boldsymbol{\epsilon} \geqslant \boldsymbol{\epsilon} : \mathbb{E}(\mathbf{x}) : \boldsymbol{\epsilon} \geqslant \alpha^{-}\boldsymbol{\epsilon} : \boldsymbol{\epsilon}$, $\infty > \alpha^{+}, \alpha^{-} > 0$, where $E_{ijkl}(\mathbf{x}) = E_{jikl}(\mathbf{x}) = E_{ijlk}(\mathbf{x}) = E_{klij}(\mathbf{x})$, $1 \leqslant i, j, k, l \leqslant 3$. The quantities R_{ijkl} and $E_{ijkl}(\mathbf{x})$ are the Cartesian components of $\mathbb{R}$ and $\mathbb{E}(\mathbf{x})$, respectively. The microstructure is assumed to be perfectly bonded. Following the standard notation, $H^1(\Omega)$ is denoted as the usual (Sobolev) space of functions with generalized partial derivatives of order $\leqslant 1$ in $L^2(\Omega)$. The symbol $\mathbf{H}^1(\Omega) \stackrel{\text{def}}{=} [H^1(\Omega)]^3$ is defined as the space of vector-valued functions whose components have generalized partial derivatives of order $\leqslant 1$ in $\mathbf{L}^2(\Omega) \stackrel{\text{def}}{=} [L^2(\Omega)]^3$.

2.1. HOMOGENEOUS-COEFFICIENT BOUNDARY-VALUE PROBLEM

The solution to the constant coefficient problem, denoted as a *regular* solution $\mathbf{u}^{\mathrm{R}}$ is characterized by a virtual work formulation:

$$\begin{aligned}&\text{Find } a\ \mathbf{u}^{\mathrm{R}} \in \mathbf{H}^1(\Omega),\ \mathbf{u}^{\mathrm{R}}|_{\Gamma_u} = \mathbf{d}, \quad \text{such that}\\ &\underbrace{\int_\Omega \nabla\mathbf{v} : \boldsymbol{\sigma}^{\mathrm{R}}\,\mathrm{d}\Omega}_{\stackrel{\text{def}}{=}\mathcal{B}^{\mathrm{R}}(\mathbf{u}^{\mathrm{R}},\mathbf{v})} = \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\Omega + \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A}_{\stackrel{\text{def}}{=}\mathcal{F}(\mathbf{v})} \quad \forall\mathbf{v}\in\mathbf{H}^1(\Omega),\ \mathbf{v}|_{\Gamma_u} = \mathbf{0},\end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}^{\mathrm{R}} = \mathbb{R} : \nabla\mathbf{u}^{\mathrm{R}}$. The equivalent complementary form is

$$\begin{aligned}&\text{Find } \boldsymbol{\sigma}^{\mathrm{R}},\ \nabla\cdot\boldsymbol{\sigma}^{\mathrm{R}} + \mathbf{f} = \mathbf{0},\ \boldsymbol{\sigma}^{\mathrm{R}}\cdot\mathbf{n}|_{\Gamma_t} = \mathbf{t} \quad \text{such that}\\ &\underbrace{\int_\Omega \boldsymbol{\tau} : \mathbb{R}^{-1} : \boldsymbol{\sigma}^{\mathrm{R}}\,\mathrm{d}\Omega}_{\stackrel{\text{def}}{=}\mathcal{A}^{\mathrm{R}}(\boldsymbol{\sigma}^{\mathrm{R}},\boldsymbol{\tau})} = \underbrace{\int_{\Gamma_u}\boldsymbol{\tau}\cdot\mathbf{n}\cdot\mathbf{d}\,\mathrm{d}A}_{\stackrel{\text{def}}{=}\mathcal{G}(\boldsymbol{\tau})} \quad \forall\boldsymbol{\tau},\ \nabla\cdot\boldsymbol{\tau} = \mathbf{0},\ \boldsymbol{\tau}\cdot\mathbf{n}|_{\Gamma_t} = \mathbf{0}.\end{aligned} \tag{2}$$

For the complementary problem, similar restrictions are imposed on the solution and test fields to force the integrals to make sense. In other words, it is assumed that solutions produce finite global energy.

2.2. INHOMOGENEOUS-COEFFICIENT BOUNDARY-VALUE PROBLEM

The solution corresponding to a material with microstructure is $\mathbf{u}$, and is characterized by the following virtual work formulation:

$$\begin{aligned}&\text{Find } \mathbf{u} \in \mathbf{H}^1(\Omega),\ \mathbf{u}|_{\Gamma_u} = \mathbf{d}, \quad \text{such that}\\ &\underbrace{\int_\Omega \nabla\mathbf{v} : \boldsymbol{\sigma}\,\mathrm{d}\Omega}_{\stackrel{\text{def}}{=}\mathcal{B}(\mathbf{u},\mathbf{v})} = \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\Omega + \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A}_{\stackrel{\text{def}}{=}\mathcal{F}(\mathbf{v})} \quad \forall\mathbf{v}\in\mathbf{H}^1(\Omega),\ \mathbf{v}|_{\Gamma_u} = \mathbf{0}.\end{aligned} \tag{3}$$

In the infinitesimal strain linearly elastic case, then $\boldsymbol{\sigma} = \mathbb{E} : \nabla \mathbf{u}$. The equivalent complementary form is

$$\text{Find } \boldsymbol{\sigma},\ \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0},\ \boldsymbol{\sigma} \cdot \mathbf{n}|_{\Gamma_t} = \mathbf{t} \quad \text{such that}$$
$$\underbrace{\int_\Omega \boldsymbol{\tau} : \mathbb{E}^{-1} : \boldsymbol{\sigma}\, \mathrm{d}\Omega}_{\stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\tau})} = \underbrace{\int_{\Gamma_u} \boldsymbol{\tau} \cdot \mathbf{n} \cdot \mathbf{d}\, \mathrm{d}A}_{\stackrel{\text{def}}{=} \mathcal{G}(\boldsymbol{\tau})} \quad \forall \boldsymbol{\tau},\ \nabla \cdot \boldsymbol{\tau} = \mathbf{0},\ \boldsymbol{\tau} \cdot \mathbf{n}|_{\Gamma_t} = \mathbf{0}. \tag{4}$$

3. Homogeneous/Inhomogeneous Solution Difference Bounds

A classical result is that, for any kinematically admissible function $\mathbf{w}$, using the (bilinear operator) notation in equation (3), a definition of the so-called energy norm is

$$\begin{aligned} 0 \ &\leqslant\ \|\mathbf{u} - \mathbf{w}\|^2_{E(\Omega)} \stackrel{\text{def}}{=} \mathcal{B}(\mathbf{u} - \mathbf{w}, \mathbf{u} - \mathbf{w}) \\ &=\ 2\mathcal{J}(\mathbf{w}) - 2\mathcal{J}(\mathbf{u}) \Rightarrow \mathcal{J}(\mathbf{u}) \leqslant \mathcal{J}(\mathbf{w}), \end{aligned} \tag{5}$$

where the "elastic potential" is defined by $\mathcal{J}(\mathbf{w}) \stackrel{\text{def}}{=} \frac{1}{2}\mathcal{B}(\mathbf{w}, \mathbf{w}) - \mathcal{F}(\mathbf{w})$. The potential relation in equation (5) is a form of the Principle of Minimum Potential Energy. Similarly the Principle of Minimum Complementary Energy, for any statically admissible field $\boldsymbol{\gamma}$, is

$$\begin{aligned} 0 \ &\leqslant\ \|\boldsymbol{\sigma} - \boldsymbol{\gamma}\|^2_{E^{-1}(\Omega)} = \mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\gamma}, \boldsymbol{\sigma} - \boldsymbol{\gamma}) \\ &=\ 2\mathcal{K}(\boldsymbol{\gamma}) - 2\mathcal{K}(\boldsymbol{\sigma}) \Rightarrow \mathcal{K}(\boldsymbol{\sigma}) \leqslant \mathcal{K}(\boldsymbol{\gamma}), \end{aligned} \tag{6}$$

where $\mathcal{K}(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \frac{1}{2}\mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - \mathcal{G}(\boldsymbol{\gamma})$. By adding the potential energy and the complementary energy of the inhomogeneous solution field, one obtains an equation of energy balance: $\mathcal{J}(\mathbf{u}) + \mathcal{K}(\boldsymbol{\sigma}) = 0$. The preceeding well-known results are found in, for example, [6]. The objective is now to bound $\|\mathbf{u} - \mathbf{u}^{\mathrm{R}}\|_{E(\Omega)}$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{R}}\|_{E^{-1}(\Omega)}$ in terms of $\mathbb{R}$, $\mathbb{E}$, $\mathbf{u}^{\mathrm{R}}$ and $\boldsymbol{\sigma}^{\mathrm{R}}$.

3.1. PRIMAL AND COMPLEMENTARY BOUNDS

Directly from (1) and (3), one has

$$\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla \mathbf{u}\, \mathrm{d}\Omega = \int_\Omega \nabla \mathbf{v} : \mathbb{R} : \nabla \mathbf{u}^{\mathrm{R}}\, \mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}\, \mathrm{d}A. \tag{7}$$

Subtracting $\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla \mathbf{u}^{\mathrm{R}}\, \mathrm{d}\Omega$ from both sides yields

$$\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla\big(\mathbf{u} - \mathbf{u}^{\mathrm{R}}\big)\, \mathrm{d}\Omega = \int_\Omega \nabla \mathbf{v} : \big(\mathbb{R} : \nabla \mathbf{u}^{\mathrm{R}} - \mathbb{E} : \nabla \mathbf{u}^{\mathrm{R}}\big) \mathrm{d}\Omega. \tag{8}$$

Since $\mathbf{v}$ is an arbitrary virtual displacement, one may set $\mathbf{v} = \mathbf{u} - \mathbf{u}^{\mathrm{R}}$ to yield

$$\begin{aligned}
\left\|\mathbf{u} - \mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)} &= \int_\Omega \nabla\big(\mathbf{u} - \mathbf{u}^{\mathrm{R}}\big) : \big(\mathbb{R} : \nabla\mathbf{u}^{\mathrm{R}} - \mathbb{E} : \nabla\mathbf{u}^{\mathrm{R}}\big)\mathrm{d}\Omega \\
&= \int_\Omega \big(\mathbb{E}^{1/2} : \nabla(\mathbf{u} - \mathbf{u}^{\mathrm{R}})\big) : \mathbb{E}^{1/2} : \mathbb{E}^{-1} : \big((\mathbb{R} - \mathbb{E}) : \nabla\mathbf{u}^{\mathrm{R}}\big)\mathrm{d}\Omega \\
&\leqslant \underbrace{\left(\int_\Omega \nabla\big(\mathbf{u} - \mathbf{u}^{\mathrm{R}}\big) : \mathbb{E} : \nabla\big(\mathbf{u} - \mathbf{u}^{\mathrm{R}}\big)\mathrm{d}\Omega\right)^{1/2}}_{\|\mathbf{u}-\mathbf{u}^{\mathrm{R}}\|_{E(\Omega)}} \\
&\quad \times \left(\int_\Omega \big((\mathbb{R} - \mathbb{E}) : \nabla\mathbf{u}^{\mathrm{R}}\big) : \mathbb{E}^{-1} : \big((\mathbb{R} - \mathbb{E}) : \nabla\mathbf{u}^{\mathrm{R}}\big)\mathrm{d}\Omega\right)^{1/2}.
\end{aligned} \tag{9}$$

Therefore, a primal solution difference upper bound is

$$\left\|\mathbf{u} - \mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)} \leqslant \underbrace{\int_\Omega \big((\mathbb{R} - \mathbb{E}) : \nabla\mathbf{u}^{\mathrm{R}}\big) : \mathbb{E}^{-1} : \big((\mathbb{R} - \mathbb{E}) : \nabla\mathbf{u}^{\mathrm{R}}\big)\mathrm{d}\Omega}_{\stackrel{\mathrm{def}}{=}(\mathcal{H}^{+,u^{\mathrm{R}}})^2}. \tag{10}$$

To derive this result, the Cauchy–Schwarz inequality was used, as well as the fact that a symmetric positive-definite matrix has a unique square root, $\mathbb{E} = \mathbb{E}^{1/2} : \mathbb{E}^{1/2}$. *The $\mathcal{H}^{+,u^{\mathrm{R}}}$-bound depends only on* $\mathbb{R}$, $\mathbb{E}$, *and* $\mathbf{u}^{\mathrm{R}}$. The process is repeated in a similar manner to form a complementary upper bound:

$$\left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)} \leqslant \underbrace{\int_\Omega \big(\big(\mathbb{R}^{-1} - \mathbb{E}^{-1}\big) : \boldsymbol{\sigma}^{\mathrm{R}}\big) : \mathbb{E} : \big(\big(\mathbb{R}^{-1} - \mathbb{E}^{-1}\big) : \boldsymbol{\sigma}^{\mathrm{R}}\big)\mathrm{d}\Omega}_{\stackrel{\mathrm{def}}{=}(\mathcal{H}^{+,\sigma^{\mathrm{R}}})^2}. \tag{11}$$

If one chooses $\mathbf{w} = \mathbf{u}^{\mathrm{R}}$ in equation (5), which is a kinematically admissible function, one has

$$\mathcal{J}(\mathbf{u}) = \mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big) - \frac{1}{2}\left\|\mathbf{u} - \mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)}.$$

Correspondingly, choosing $\boldsymbol{\gamma} = \boldsymbol{\sigma}^{\mathrm{R}}$, one has from equation (6),

$$\mathcal{K}(\boldsymbol{\sigma}) = \mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big) - \frac{1}{2}\left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)}.$$

Combining the results we obtain an "exact" bound on a hybrid norm

$$\underbrace{\big(\mathcal{T}\big(\mathbf{u}^{\mathrm{R}}, \boldsymbol{\sigma}^{\mathrm{R}}\big)\big)^2 \stackrel{\mathrm{def}}{=} 2\big(\mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big) + \mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big)\big)}_{\text{HYBRID BOUND}} = \left\|\mathbf{u} - \mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)} + \left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)}, \tag{12}$$

since $\mathcal{J}(\mathbf{u}) + \mathcal{K}(\boldsymbol{\sigma}) = 0$.

From equation (12), lower primal and complementary bounds follow,

$$\begin{aligned}
\left\|\mathbf{u}-\mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)} &= 2\big(\mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big)+\mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big)\big)-\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)},\\
&\geqslant 2\big(\mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big)+\mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big)\big)-\big(\mathcal{H}^{+,\sigma^{\mathrm{R}}}\big)^2 \stackrel{\text{def}}{=} \big(\mathcal{H}^{-,u^{\mathrm{R}}}\big)^2,\\
\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)} &= 2\big(\mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big)+\mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big)\big)-\left\|\mathbf{u}-\mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)},\\
&\geqslant 2\big(\mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big)+\mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big)\big)-\big(\mathcal{H}^{+,u^{\mathrm{R}}}\big)^2 \stackrel{\text{def}}{=} \big(\mathcal{H}^{-,\sigma^{\mathrm{R}}}\big)^2.
\end{aligned} \tag{13}$$

Therefore, in summary

$$\begin{aligned}
\big(\mathcal{H}^{-,u^{\mathrm{R}}}\big)^2 &\leqslant \underbrace{2\big(\mathcal{J}\big(\mathbf{u}^{\mathrm{R}}\big)-\mathcal{J}(\mathbf{u})\big)}_{\underbrace{\|\mathbf{u}-\mathbf{u}^{\mathrm{R}}\|^2_{E(\Omega)}}_{\text{SOLUTION DIFFERENCE}}} \leqslant \big(\mathcal{H}^{+,u^{\mathrm{R}}}\big)^2,\\
\big(\mathcal{H}^{-,\sigma^{\mathrm{R}}}\big)^2 &\leqslant \underbrace{2\big(\mathcal{K}\big(\boldsymbol{\sigma}^{\mathrm{R}}\big)-\mathcal{K}(\boldsymbol{\sigma})\big)}_{\underbrace{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{R}}\|^2_{E^{-1}(\Omega)}}_{\text{SOLUTION DIFFERENCE}}} \leqslant \big(\mathcal{H}^{+,\sigma^{\mathrm{R}}}\big)^2,\\
\big(\mathcal{T}\big(\mathbf{u}^{\mathrm{R}},\boldsymbol{\sigma}^{\mathrm{R}}\big)\big)^2 &= \left\|\mathbf{u}-\mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)}+\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)}.
\end{aligned} \tag{14}$$

These relations hold for any loading, external geometry, and a pointwise positive-definite microstructure.

3.2. A SPECIAL CASE

Consider the special case of a microstructure composed of a matrix embedded with particulate matter. The expressions attain very compact forms. The heterogeneous material can be represented as $\mathbb{E}=\mathbb{R}+\Delta\mathbb{R}$ or $\mathbb{E}^{-1}=\mathbb{R}^{-1}+\Delta\mathbb{R}^{-1}$, where, in general, $\Delta\mathbb{R}^{-1}\neq(\Delta\mathbb{R})^{-1}$. Referring to Figure 1, it is clear that $\Delta\mathbb{R}=\mathbb{E}-\mathbb{R}=\mathbf{0}$ in the matrix. Therefore, the upper bounds collapse to

$$\begin{aligned}
\left\|\mathbf{u}-\mathbf{u}^{\mathrm{R}}\right\|^2_{E(\Omega)} &\leqslant \sum_{i=i}^{N}\int_{\Omega_i}\big(\Delta\mathbb{R}:\nabla\mathbf{u}^{\mathrm{R}}\big):\mathbb{E}^{-1}:\big(\Delta\mathbb{R}:\nabla\mathbf{u}^{\mathrm{R}}\big)\mathrm{d}\Omega_i,\\
\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{R}}\right\|^2_{E^{-1}(\Omega)} &\leqslant \sum_{i=i}^{N}\int_{\Omega_i}\big(\Delta\mathbb{R}^{-1}:\boldsymbol{\sigma}^{\mathrm{R}}\big):\mathbb{E}:\big(\Delta\mathbb{R}^{-1}:\boldsymbol{\sigma}^{\mathrm{R}}\big)\mathrm{d}\Omega_i,
\end{aligned} \tag{15}$$

where Ω_i are the domains of the (N) inhomogeneities. In other words, the difference can be computed solely in terms of the product of the difference between the elastic properties of inhomogeneities and the matrix, and the unperturbed homogeneous elastic field. *Furthermore, this only needs to be computed over the domains of the inhomogeneities*.

4. Concluding Remarks

A possible use of such bounds is to aid in characterizing the internal effects of additives in the composite material design development. During such processes, information about the magnitude of the resulting distortion of the otherwise smooth internal fields, corresponding to the matrix material alone, may be valuable. Throughout the analysis, $\mathbb{R}$ was assumed to be the homogeneous matrix material, however, as illustrated by the proofs, this is unnecessary. The tensor $\mathbb{R}$ can also be a spatially-variable function, with no alteration in the analysis. The use of alternative definitions of $\mathbb{R}$ has been exploited for, primitive versions of the presented bounds, in [7].

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