Uncertainty growth in Hypo-Elastic Material Models

T. I. Zohdi

Mathematics and Mechanics of Solids 2006 11: 555
DOI: 10.1177/1081286505046485

Uncertainty growth in Hypo-Elastic Material Models

T. I. ZOHDI
Department of Mechanical Engineering, 6195 Etcheverry Hall, University of California, Berkeley, CA, 94720-1740, USA

(Received 22 March 2004; accepted 5 April 2004)

Abstract: The purpose of this note is to highlight a new point of concern in the use of hypo-elastic constitutive equations. In particular, it is shown that if there is any uncertainty in the material constants that appear in such equations, it can induce nonmonotone uncertainty growth in the resulting solutions.

Key Words: Hypo-elastic, uncertainty, nonmonotone growth

1. INTRODUCTION

Hypo-elasticity endeavors to describe the elastic response of a material based upon rates. The theory was developed by Truesdell [1], his motivation being to extend the classical linear elastic theory to large strains, however, without assuming the usual hyperelastic approach of *stress* $= \mathcal{F}$(*finite strain*). The approach is to write *rate of stress* $= \mathcal{F}$(*rate of deformation*), with the goal being to obtain a description of elastic behavior expressed entirely in terms of rates, however avoiding the effects of viscosity, relaxation, etc. The model serves as a starting point for a number of rate-type models in elasto-plasticity by augmenting the basic hypo-elastic model with flow rules governing the evolution of plastic strain metrics. The classical hypo-elastic approach is to write

$$\overset{\circ}{\mathbf{T}} \stackrel{\text{def}}{=} \dot{\mathbf{T}} - \mathbf{W} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{W} = \mathbf{H}(\mathbf{T}, \mathbf{D}), \tag{1.1}$$

where $\overset{\circ}{\mathbf{T}}$ is the Jaumann rate of the Cauchy stress, $\mathbf{T}$, where $\mathbf{H}(\mathbf{T}, \mathbf{D})$ is a tensor function which is linear and isotropic in the symmetric part of the velocity ($\mathbf{v}$) gradient, denoted by $\mathbf{D} \stackrel{\text{def}}{=} \frac{1}{2}(\nabla_x \mathbf{v} + (\nabla_x \mathbf{v})^T)$, and linear in $\mathbf{T}$, and where $\mathbf{W} \stackrel{\text{def}}{=} \frac{1}{2}(\nabla_x \mathbf{v} - (\nabla_x \mathbf{v})^T)$ is the vorticity tensor. (Of course, such a formulation can be extended to anisotropy by incorporating dependency on structure tensors accompanied by their own evolution laws.) For extensive mathematical details on this class of models see Truesdell [1, 2] or Truesdell and Noll [3]. Probably the simplest of member of this family of representations is

$$\overset{\circ}{\mathbf{T}} = \kappa \, \mathrm{tr}\,(\mathbf{D})\mathbf{1} + 2\mu \mathbf{D}', \tag{1.2}$$

where κ is the bulk modulus, where μ is the shear modulus, where $tr(\mathbf{D})$ is the trace of $\mathbf{D}$ and where $\mathbf{D}' = \mathbf{D} - \frac{\mathrm{tr}\,(\mathbf{D})}{3}\mathbf{1}$ is the deviatoric part of $\mathbf{D}$. There have been numerous

Mathematics and Mechanics of Solids, **11:** 555–562, 2006 DOI: 10.1177/1081286505046485

criticisms of such models, such as artificial softening, under certain conditions, which was initially recognized by Truesdell [2]. Furthermore, when such models are used in conjunction with plasticity relations, the occurance of stress oscillations in large simple elastic shear can result. This behavior has been noted by several authors, dating back, at least, to Lehmann [4] and Dienes [5]. A variety of approaches for suppression of such oscillations have been suggested, for example, by Dafalias [6]. However, a question that is usually not addressed is the sensitivity of such models to uncertainty or error in their material constants. (We shall use the terms "uncertainty" and "error" interchangeably throughout the analysis.) For example, first take a one-dimensional version of Equation (1.2):

$$\dot{T} = \mathbb{E}D, \tag{1.3}$$

where $\mathbb{E}$ is Young's modulus. Now assume that there is an uncertainty in $\mathbb{E}$:

$$\dot{\tilde{T}} = \tilde{E}D = (\mathbb{E} + \delta\mathbb{E})D, \tag{1.4}$$

where D is controlled. Subtracting Equation (1.3) from Equation (1.4) yields an equation for the error, $\phi = \tilde{T} - T$,

$$\dot{\phi} = \dot{\tilde{T}} - \dot{T} = \delta\mathbb{E}D. \tag{1.5}$$

Thus, for a constant D, and constant error in the material constant ($\delta\mathbb{E}$), a constant error results in the stress rates. This indicates that the error in stress grows in time linearly with uncertainty in the material constant, which is acceptable.

Now consider the perturbed three-dimensional model

$$\overset{\circ}{\tilde{\mathbf{T}}} = (\kappa + \delta\kappa)\,\mathrm{tr}\,(\mathbf{D})\mathbf{1} + 2(\mu + \delta\mu)\mathbf{D}', \tag{1.6}$$

where $\delta\kappa$ and $\delta\mu$ are the uncertainties in the bulk (κ) and shear moduli (μ) respectively. Subtracting Equation (1.2) from (1.6) yields an equation for the error $\boldsymbol{\phi} \overset{\text{def}}{=} \tilde{\mathbf{T}} - \mathbf{T}$, where $\mathbf{D}$ is controlled:

$$\overset{\circ}{\tilde{\mathbf{T}}} - \overset{\circ}{\mathbf{T}} = \dot{\boldsymbol{\phi}} - \mathbf{W}\cdot\boldsymbol{\phi} + \boldsymbol{\phi}\cdot\mathbf{W} = \delta\kappa\,\mathrm{tr}\,(\mathbf{D})\mathbf{1} + 2\delta\mu\mathbf{D}'. \tag{1.7}$$

Thus, the character of the operator

$$\mathbf{A}(\boldsymbol{\phi}, \mathbf{W}) \overset{\text{def}}{=} -\mathbf{W}\cdot\boldsymbol{\phi} + \boldsymbol{\phi}\cdot\mathbf{W} \tag{1.8}$$

will dictate the growth of the error due to any uncertainty in the material constants. This error growth may be exponential in time, which is unacceptable. (There are a variety of other possible detrimental effects due to uncertainty in material parameters, and the interested reader is referred to Zohdi [7–9].)

2. AN EXAMPLE

As an example, consider a plane strain problem with the following constant vorticity ($K =$ constant),

$$\mathbf{W} = K \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1}$$

and the following (symmetric) plane strain error:

$$\boldsymbol{\phi} = \tilde{\mathbf{T}} - \mathbf{T} = \begin{bmatrix} \phi_{xx} & \phi_{xy} & 0 \\ \phi_{yx} & \phi_{yy} & 0 \\ 0 & 0 & \phi_{zz} \end{bmatrix}. \tag{2.2}$$

We have the following:

$$\mathbf{A}(\boldsymbol{\phi}, \mathbf{W}) \stackrel{\text{def}}{=} -\mathbf{W} \cdot \boldsymbol{\phi} + \boldsymbol{\phi} \cdot \mathbf{W} = K \begin{bmatrix} 2\phi_{xy} & \phi_{yy} - \phi_{xx} & 0 \\ \phi_{yy} - \phi_{xx} & -2\phi_{xy} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

Thus, we have three coupled equations:

$$\dot{\phi}_{xx} + 2K\phi_{xy} = \left(\delta\kappa + \frac{4\delta\mu}{3}\right) D_{xx} + \left(\delta\kappa - \frac{2\delta\mu}{3}\right) D_{yy}, \tag{2.4}$$

and

$$\dot{\phi}_{yy} - 2K\phi_{xy} = \left(\delta\kappa - \frac{2\delta\mu}{3}\right) D_{xx} + \left(\delta\kappa + \frac{4\delta\mu}{3}\right) D_{yy}, \tag{2.5}$$

and

$$\dot{\phi}_{xy} + K(\phi_{yy} - \phi_{xx}) = 2\delta\mu D_{xy}. \tag{2.6}$$

We remark that the equation for the z direction is uncoupled from the other three equations and reads as ($D_{zz} = 0$)

$$\dot{\phi}_{zz} = \left(\delta\kappa - \frac{2\delta\mu}{3}\right)(D_{xx} + D_{yy}). \tag{2.7}$$

In matrix form, the coupled system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{xx} \\ \dot{\phi}_{yy} \\ \dot{\phi}_{xy} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2K \\ 0 & 0 & -2K \\ -K & K & 0 \end{bmatrix} \begin{Bmatrix} \phi_{xx} \\ \phi_{yy} \\ \phi_{xy} \end{Bmatrix}$$

$$= \begin{Bmatrix} (\delta\kappa + \frac{4\delta\mu}{3})D_{xx} & + (\delta\kappa - \frac{2\delta\mu}{3})D_{yy} \\ (\delta\kappa - \frac{2\delta\mu}{3})D_{xx} & + (\delta\kappa + \frac{4\delta\mu}{3})D_{yy} \\ & 2\delta\mu\, D_{xy} \end{Bmatrix}. \tag{2.8}$$

The eigenvalues of the coupling matrix are $\lambda_1 = 2Ki$, $\lambda_2 = -2Ki$ and $\lambda_3 = 0$. The corresponding eigenvectors are

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \end{bmatrix}_{\lambda=2Ki}, \quad \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} \end{bmatrix}_{\lambda=-2Ki}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}_{\lambda=0}. \tag{2.9}$$

Performing a similarity transform to decouple the system, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\hat{\phi}}_{xx} \\ \dot{\hat{\phi}}_{yy} \\ \dot{\hat{\phi}}_{xy} \end{bmatrix} + \begin{bmatrix} 2Ki & 0 & 0 \\ 0 & -2Ki & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{\phi}_{xx} \\ \hat{\phi}_{yy} \\ \hat{\phi}_{xy} \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 0 \end{bmatrix}^{-1}$$

$$\times \begin{Bmatrix} (\delta\kappa + \frac{4\delta\mu}{3})D_{xx} & + (\delta\kappa - \frac{2\delta\mu}{3})D_{yy} \\ (\delta\kappa - \frac{2\delta\mu}{3})D_{xx} & + (\delta\kappa + \frac{4\delta\mu}{3})D_{yy} \\ & 2\delta\mu\, D_{xy} \end{Bmatrix} \stackrel{\text{def}}{=} \begin{Bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{Bmatrix}. \tag{2.10}$$

The decoupled problems can be written as

$$\dot{\hat{\phi}}_{xx} + 2Ki\hat{\phi}_{xx} = \hat{f}_1, \tag{2.11}$$

$$\dot{\hat{\phi}}_{yy} - 2Ki\hat{\phi}_{yy} = \hat{f}_2, \tag{2.12}$$

$$\dot{\hat{\phi}}_{xy} = \hat{f}_3, \tag{2.13}$$

and can be solved individually to yield

$$\hat{\phi}_{xx}(t) = \left(\hat{\phi}_{xx}(0) - \frac{\hat{f}_1}{2Ki}\right) e^{-2Kit} + \frac{\hat{f}_1}{2Ki}, \quad (2.14)$$

$$\hat{\phi}_{yy}(t) = \left(\hat{\phi}_{yy}(0) + \frac{\hat{f}_2}{2Ki}\right) e^{2Kit} - \frac{\hat{f}_2}{2Ki}, \quad (2.15)$$

$$\hat{\phi}_{xy}(t) = \hat{f}_3 t + \hat{\phi}_{xy}(0). \quad (2.16)$$

Afterwards, the solutions are transformed back to yield

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 0 \end{bmatrix} \begin{Bmatrix} \hat{\phi}_{xx} \\ \hat{\phi}_{yy} \\ \hat{\phi}_{xy} \end{Bmatrix} = \begin{Bmatrix} \phi_{xx} \\ \phi_{yy} \\ \phi_{xy} \end{Bmatrix}. \quad (2.17)$$

We have the following observations:

- The presence of the complex-valued exponential terms throughout the solution system will lead to oscillatory behavior of the solution uncertainty, in addition to super-imposed linear growth.
- The magnitude of K controls the rapidity of the solution uncertainty oscillations.
- The magnitude of terms like $\left(\hat{\phi}_{xx}(0) - \frac{\hat{f}_1}{2Ki}\right)$ and $\left(\hat{\phi}_{yy}(0) + \frac{\hat{f}_2}{2Ki}\right)$ control the the amplitude of the oscillations. When such terms are large, their amplitudes become large. When K is increased, the influence of terms like $\frac{\hat{f}_1}{2Ki}$ diminishes.
- The transformation back to the original coordinates eliminates any linear growth for the $\phi_{xy}(t)$ term. The solution uncertainty components can be written as

  $$\phi_{xx}(t) = a_1 \cos(2Kt) + a_2 \sin(2Kt) + a_3 t + a_4, \quad (2.18)$$

  and

  $$\phi_{yy}(t) = b_1 \cos(2Kt) + b_2 \sin(2Kt) + b_3 t + b_4, \quad (2.19)$$

  and

  $$\phi_{xy}(t) = c_1 \cos(2Kt) + c_2 \sin(2Kt) + c_3, \quad (2.20)$$

  where the constants, which are somewhat lengthy, can be explicitly expressed in terms of the loading an initial conditions by equating the complex and real parts of the transformed solution.

As as example consider the following material perturbations $\delta\kappa = 100$ Mpa and $\delta\mu = 100$ Mpa and four different values of the rate constant appearing in the vorticity (1) $K = 0.005 s^{-1}$, (2) $K = 0.01 s^{-1}$, (3) $K = 0.025 s^{-1}$ and (4) $K = 0.05 s^{-1}$. The

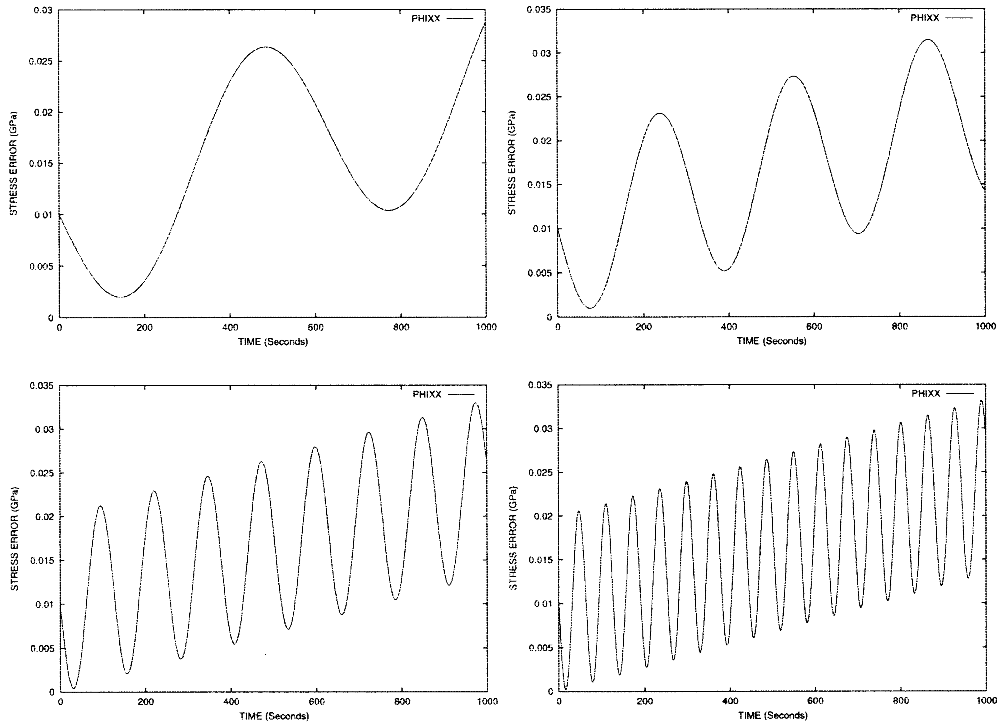

Figure 1. Starting from left to right and top to bottom, the growth in the ϕ_{xx} (stress uncertainty) component due to material perturbations under displacement control for (1) $K = 0.005\,\mathrm{s}^{-1}$, (2) $K = 0.01\,\mathrm{s}^{-1}$, (3) $K = 0.025\,\mathrm{s}^{-1}$ and (4) $K = 0.05\,\mathrm{s}^{-1}$.

components of $\mathbf{D}$ were all set to zero, except for $D_{xx} = 0.001\,s^{-1}$. The initial uncertainties in the components were set to $\phi_{xx}(0) = 0.01$ Gpa, $\phi_{yy}(0) = 0.01$ GPa and $\phi_{xy}(0) = 0.01$ Gpa. The growth of the solution uncertainty is linear with superimposed oscillations, as illustrated in Figures 1–3.

3. GENERALIZATIONS

Clearly, the previous results provide a criteria by which to choose a general $\mathbf{H}(\mathbf{T}, \mathbf{D})$ to mitigate the effects of uncertainty. For this more general case, consider

$$\overset{\circ}{\mathbf{T}} = \mathbf{H}(\mathbf{T}, \mathbf{D}), \tag{3.1}$$

and the perturbed operator, operating on the perturbed Cauchy stress

$$\overset{\circ}{\tilde{\mathbf{T}}} = \tilde{\mathbf{H}}(\tilde{\mathbf{T}}, \mathbf{D}). \tag{3.2}$$

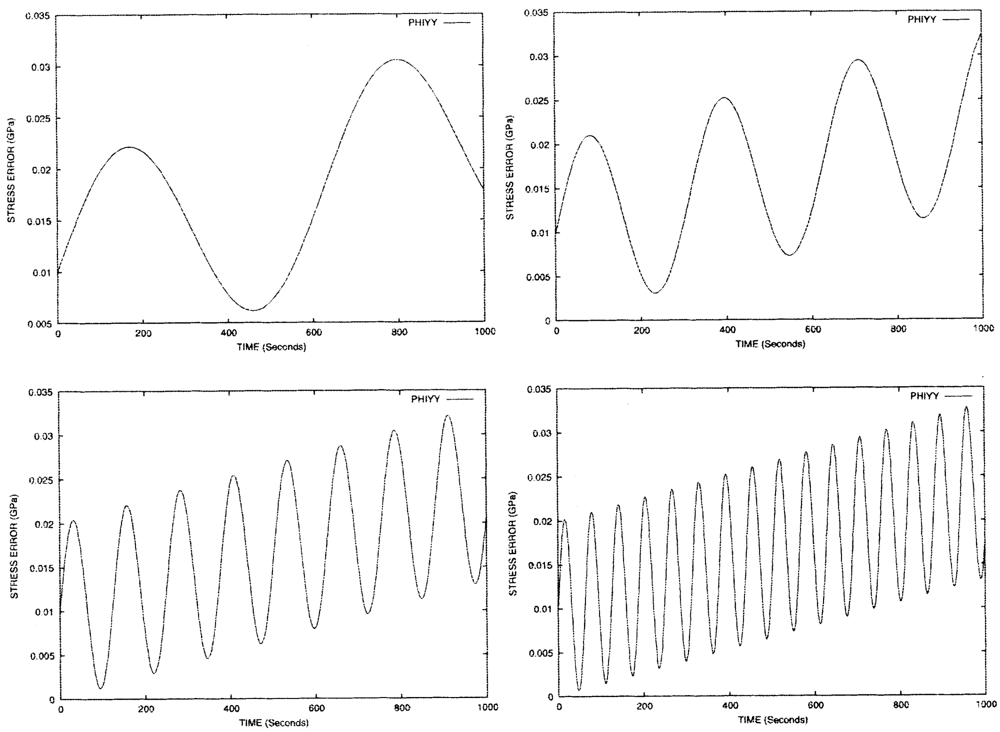

Figure 2. Starting from left to right and top to bottom, the growth in the ϕ_{yy} (stress uncertainty) component due to material perturbations under displacement control for (1) $K = 0.005\,s^{-1}$, (2) $K = 0.01\,s^{-1}$, (3) $K = 0.025\,s^{-1}$ and (4) $K = 0.05\,s^{-1}$.

Subtracting yields

$$\overset{\circ}{\tilde{\mathbf{T}}} - \overset{\circ}{\mathbf{T}} = \dot{\boldsymbol{\phi}} - \mathbf{W}\cdot\boldsymbol{\phi} + \boldsymbol{\phi}\cdot\mathbf{W} = \tilde{\mathbf{H}}(\tilde{\mathbf{T}},\mathbf{D}) - \mathbf{H}(\mathbf{T},\mathbf{D}) \stackrel{\text{def}}{=} \mathbf{B}(\mathbf{T},\tilde{\mathbf{T}},\mathbf{D}). \tag{3.3}$$

Thus, we have

$$\dot{\boldsymbol{\phi}} + \mathbf{A}(\boldsymbol{\phi},\mathbf{W}) = \mathbf{B}(\mathbf{T},\tilde{\mathbf{T}},\mathbf{D}), \tag{3.4}$$

which governs the behavior of the error. Thus, for the general hypo-elastic case, one could construct $\mathbf{H}(\mathbf{T},\mathbf{D})$, by proper selection of $\mathbf{A}$ and $\mathbf{B}$, to be as insensitive as possible to material perturbations, provided that the construction does not conflict with other restrictions, for example such as those cited in Casey and Naghdi [10]. However, generally, this may not be a simple task.

Acknowledgements. The author wishes to thank his colleagues James Casey and David Steigmann for their constructive comments during the preparation of this manuscript. The author also wishes to thank Prof. Alan Needleman for pointing out an important algebraic error in an earlier version of the manuscript.

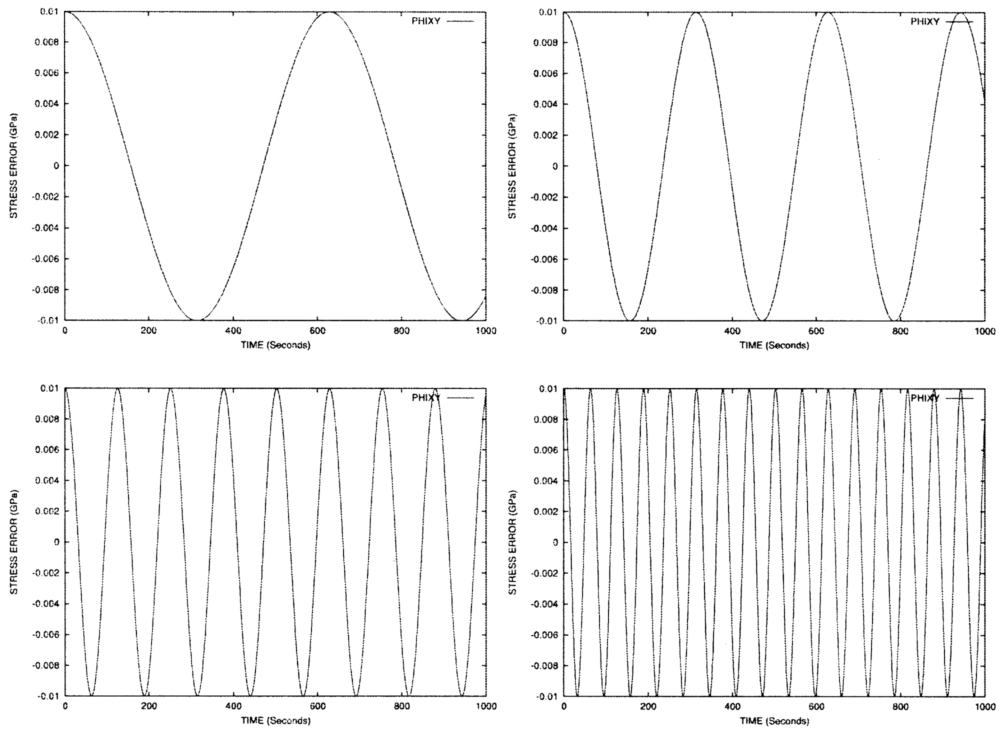

Figure 3. Starting from left to right and top to bottom, the growth in the ϕ_{xy} (stress uncertainty) component due to material perturbations under displacement control for (1) $K = 0.005\,s^{-1}$, (2) $K = 0.01\,s^{-1}$, (3) $K = 0.025\,s^{-1}$ and (4) $K = 0.05\,s^{-1}$.

REFERENCES

[1] Truesdell, C. Hypo-elasticity. *Journal of Rational and Mechanical Analysis,* 4, 83–133, 1019–1020 (1955). Reprinted in Foundations of Elasticity Theory, International Science Series, Vols 99, 100, 103, Gordon and Breach, New York, 1965.

[2] Truesdell, C. Hypo-elastic shear. *Journal of Applied Physics,* 27, 441-447 (1956).

[3] Truesdell, C. and Noll, W. The nonlinear field theories, in *Handbuch der Physik,* ed. S. Flügge, Vol. 111/3, Springer, Berlin, 1965.

[4] Lehmann, T. Anisotrope plastische Formänderungen. *Revue Roumaine des Sciences Techniques. Serie de Mécanique Appliqée,* 17, 1077–1086 (1972).

[5] Dienes, J. K. On the analysis of rotation and stress rate in deforming bodies. *Acta Mechanica,* 32, 217–232 (1979).

[6] Dafalias, Y. F. Corotational rates for kinematic hardening at large plastic deformations. *Journal of Applied Mechanics,* 50, 561–565 (1983).

[7] Zohdi, T. I. On the propagation of microscale material uncertainty in a class of hyperelastic finite deformation stored energy functions. *International Journal of Fracture/Letters in Micromechanics,* 112, L13–L17 (2001).

[8] Zohdi, T. I. Bounding envelopes in multiphase material design. *Journal of Elasticity,* 66, 47–62 (2002).

[9] Zohdi, T. I. Genetic optimization of statistically uncertain microheterogeneous solids. *Philosophical Transactions of the Royal Society,* 361, 1021–1043 (2003).

[10] Casey, J. and Naghdi, P. M. On the relationship between the Eulerian and Lagrangian descriptions of finite rigid plasticity. *Archive for Rational Mechanics and Analysis,* 102, 351–375 (1988).